N GR-06-003-075

INFLUENCE OF THE AIRCRAFT SHAPE ON THE WAVE PATTERN

by

Adolf Busemann

A lecture delivered on December 12, 1968 for the Space Institute course on "Generation and Propagation of Shock Waves with Application to Sonic Boom" at the University of Tennessee, Tullahoma, Tennessee. This lecture was based in part on research supported by the National Aeronautics and Space Administration.

Department of Aerospace Engineering Sciences
University of Colorado
Boulder, Colorado 80302

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by A. Busemann (University of Colorado)

I. Introduction. The sonic boom (or bang) as a side effect of flight with supersonic speed was observed by surprise and was finally attributed to the simple straight flight itself only after all other more plausible explanations could not be supported any longer. The lesson to be learned from this strange history is the fact that the supersonic aircraft has indeed a surprisingly low attenuation of its pressure field with distance. Being used to an attenuation proportional to the minus second power of the distance in subsonic flight, it was hard to accept that at supersonic flight only a minus one half power of the lateral distance remains. Furthermore in a stratified atmosphere with density changes following an exponential law of the altitude, another influence has to be considered by the fact, that the disturbance velocities created at high altitudes diminish on the way toward the ground, whereas the accompanying pressures have to do the opposite, each taking half of the burden to adapt the disturbances to the changing elastic modulus of the denser air. As long as the original disturbance is due to invariant velocities, --i.e., by pulling the aircraft fuselage through the surrounding air--the great advantage to move in lower density at high altitudes is only half lost by the increasing pressures on the vertical descent to the earth surface. If, however, the heavier-than-air craft has to create lift with its wings, the weight distributed over the wing area represents an invariant pressure regardless of altitude. Now the acoustical pressure increase with half the ratio of the density increase is a powerful rival to the meager pressure relief with distance from the airplane. The minus one half power of the distance is strongest very early and diminishes fast with distance, while the exponential law stays constant. Thus in a finite distance equal to the scale height of the atmosphere of about 28,000 feet any additional climbing to a higher altitude is not only reaching a larger area of listeners on the ground but does not reduce the pressure footprint for the people directly underneath the flight path in the air. Only the help of nature to distort the footprint (by a faster propagation of the higher pressures and a slower propagation of the lower pressures within the disturbance) spreads the footprint over a larger distance with reduced pressure differences under the action of shock waves. This relief by shock waves is not limited to an optimal altitude and is the only advantage left after climbing above 28,000 feet. But when shocks themselves are regarded as undesirable acoustical phenomena for our ears by exciting the slumbering high frequencies which serve commonly as danger signals, the help of nature is rejected and the aforementioned reasons to reduce the altitude get even more support as a means to avoid large distortions of the footprint, the causes for shockwaves.

II. Methods of Flow Representation. The methods to study the far field pressures with respect to sonic booms are supported by the fact that there exists already a very low relative pressure of the order of one thousandth of the atmosphere setting the borderline between permissible and not permissible disturbances. Up to such a limit the linearized flow representation goes a long way to furnish the far field pressures with a satisfactory accuracy. The actual body shapes responsible for the far field pattern may require a higher order correction especially for axially symmetric cross sections because of the much stronger disturbances in the near field. Such corrections necessary for the actual design of the aircraft may, however, be disregarded in the game to reduce the sonic boom as long as all significant constraints, i.e., to use an airplane of given length and of a given average cross section distribution of positive definite size over this length, are met.

The integration of a simplified differential equation very far out has, of course, its own dangers; but the only significant negligence, which would disappear in restricted pressure amplitudes, is the fact that the wave speed of positive disturbance pressures is slightly higher, of negative disturbance pressures slightly lower than the speed of sound for infinitesimal disturbances. The effects of this self-propulsion of any given linearized footprint can be handled separately simply by reading the final results with a slanted ordinate axis, which advances every pressure toward the front of the wave proportional to its deviation from the static pressure. As is visible in water waves at the beaches and used by surf board riders for pleasure, too much advance of the higher wave portions leads to overhanging cliffs and requires an adjustment by a steep front at that position, where the overhang and the undercut areas equalize. This simple graphical equalization--well known from the van der Waals two phase formation in liquifying vapors--is the only repair required on the linearized footprint to produce the true wave pattern for the observer on the ground. That the reflection on a solid ground doubles all disturbance pressures without time delay, while a reflected wave with some delay would be audible at higher elevation is another element to consider for interpreting the far field pressure disturbance correctly.

III. Sources and Sinks in Three-Dimensional Space. To describe the flow field around moving bodies in three dimensions, the method of distributing sources and sinks inside the body while artificially extending the flow to fill the complete interior is used successfully from the time of the airship's flying in incompressible fluid and is still the most common approach at all speeds. While for incompressible flow, where superposition is absolutely correct, the source and sink method is universally valid, at compressible flow it has to be restricted to slender bodies. Such restriction is irrelevant as long as the aircraft of minimum drag (and not a reentering spacecraft) is foremost the object of our investigations.

The two basic laws of fluid motion past streamlined bodies are vanishing divergence and vanishing curl applied both to the velocity field for incompressible flow. In two dimensions both these laws have an equal amount of information and their first integrals, the stream function or the flow potential are on the same level, as their complex combination in conformal mapping reveals. For boundary conditions the streamlines are very often superior. In three dimensions the vanishing curl rates twice as high as the vanishing divergence and the flow potential as its first integral has no comparable alternate; thus it is the velocity potential which always serves as the first step toward any desired solution. Working with sources and sinks as the preferred singularities instead of vortices, puts, however, more emphasis on the flux density of the flow which, at least in steady flow, has vanishing divergence outside the body. To be able to work with the most appropriate tools in three-dimensional flow, we have to learn to "speak" in velocity potentials and to "think" in flow densities at the same time. Such language is developed for many vector fields with concrete point singularities. The most common and the earliest in history is perhaps the gravity field around a single point mass. Here the gravity potential is found proportional to the inverse distance $1/r$ from the center of the attracting mass G. The proper value of the potential energy for any unit mass entering the field is given by:

$$\Phi = -\frac{G}{4\pi r} \tag{1}$$

The actual force f felt on the unit mass is the (negative!) increase of the potential energy under radial motion:

$$f = -\frac{d\Phi}{dr} = -\frac{G}{4\pi r^2} \tag{2}$$

Working with potentials but thinking in fluxes is indicated by the flux density f, when G is considered as the total flux which must be equally spread over all concentric spheres, each having the surface $4\pi r^2$. The mass $-G$ is in this case a sink for the inwardly directed flux densities f. The results for a gravity field being a force field of nothing but attracting positive masses will change some sign, when being adapted to repelling electric charges of equal sign or attracting charges of opposite sign. Furthermore, the velocity potential which has no significance in the form of potential energy, furnishes by convention the field vector by taking its positive gradient (instead of the negative as in equation (2)). It is, therefore, the form not the signs of equations (1) and (2) which sets the pattern. A still more disturbing fact in compressible flow is the difference between the velocity vector U and the flux density vector $f = \varrho_\infty U$. Both are equal in direction; but our experience in the deLaval nozzle indicates that they are only proportional at very low speeds. When nearing the velocity of sound, the flux density reaches its maximum value at the speed of sound, while it shrinks back to the lower values at supersonic speeds in spite of the steady increase of the velocity toward the finite maximum velocity for which the nozzle is designed.

The law of superposition on a parallel flow is only valid without adaptation in incompressible flow. Compressible flow allows for small disturbances to be superimposed and requires a slight adaptation to the Mach number of the given parallel flow. Let U be the velocity of the main parallel flow with a_∞ the velocity of sound and p_∞ the static pressure far away from any disturbance. The coordinate system x,y,z may be chosen to coincide on its origin with the point source of the flux intensity S and with the x axis parallel to the undisturbed flow. While the isotropy of the source flow in all directions may be lost at larger Mach numbers, the axial symmetry with respect to the x axis should be preserved.

Such distortion of the potential spheres to ellipsoides of constant potential is suggested by the form of the linearized differential equation for the superimposed potential φ :

$$(1-M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{3}$$

But the new potential should also be normalized by a special power of $(1 - M^2)$ as a factor to indicate always the same mass flux S independent of the Mach number of the flow. The normalized equation is given by

$$\varphi = -\frac{S}{4\pi\rho_\infty\sqrt{x^2 + (1-M^2)(y^2+z^2)}} \tag{4}$$

The disturbance velocities u,v,w superimposed on the main flow U are the partial derivatives of φ with respect to x,y,z. But the disturbance flux densities f_x, f_y, and f_z superimposed on the main flux density $F = \rho_\infty U$ are ρ_∞ times the disturbance velocities only in the new directions f_y and f_z. In the main direction the example of the deLaval nozzle indicates a necessary factor $(1 - M^2)$ times ρ_∞ to take care of the change in density by a change in the length of the total velocity vector $U + u$ to $F + f_x$. The corresponding equations for the superimposed flux densities are, therefore:

$$\begin{aligned} f_x &= (1-M^2)\rho_\infty\frac{\partial\varphi}{\partial x} \\ f_y &= \rho_\infty\frac{\partial\varphi}{\partial y} \\ f_z &= \rho_\infty\frac{\partial\varphi}{\partial z} \end{aligned} \tag{5}$$

The actual differentiations, if carried out on the source potential of equation (4), furnish the following components:

$$f_x = \frac{(1-M^2)\, S\, x}{4\pi\left[x^2 + (1-M^2)(y^2+z^2)\right]^{3/2}}$$

$$f_y = \frac{(1-M^2)\, S\, y}{4\pi\left[x^2 + (1-M^2)(y^2+z^2)\right]^{3/2}} \qquad (6)$$

$$f_z = \frac{(1-M^2)\, S\, z}{4\pi\left[x^2 + (1-M^2)(y^2+z^2)\right]^{3/2}}$$

The source flow character is immediately visible by the radial flux directions away from the source point and the intensities indicate at subsonic speeds a reduced flow by $(1 - M^2)$ along the x axis compensated by increases with $\sqrt{1 - M^2}$ along the y and the z axes. The total source strength is found to be invariant, though degenerating toward a complete lateral discharge for the Mach number one.

IV. Supersonic Source Behavior in Three Dimensions. The factors $(1 - M^2)$ in the equations (4) and (6) for the three dimensional source flow appear very harmless while they render the source strength invariant for any subsonic Mach number. There is no complete change to imaginary values at supersonic speeds except for the "neutral" region outside the Mach cones, where there should be no discharge of a supersonic source in the first place. To carry over the symmetry with respect to $\pm$ x direction from subsonic flow is a possible but not a logical choice, since the only enforceable discharge of any supersonic source is within the aft cone, not the forward cone, neither half of the source intensity in each of them. A small improvement, by human decision not by analytical continuation, is to restrict the discharge to $x > 0$ while doubling the discharge rate and by using real parts of the complex potentials only. Such revised source strength distributed over any chosen region with the coordinates ξ, η, and ζ inside the body walls furnishes the following integral equation for the resulting velocity potential:

$$\varphi(x,y,z) = -\iiint \frac{s(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta}{2\pi\, \mathrm{Re}\sqrt{(x-\xi)^2 - (M^2-1)\left[(y-\eta)^2 + (z-\zeta)^2\right]}} \qquad (7)$$

The change to the lower case $s(\xi,\eta,\zeta)$ from S indicates the restriction to smaller distributed sources in order to satisfy the small disturbance requirement. This potential proposed by von Karman in the thirties satisfies the linearized differential equation, is adapted to the logical dependence regions of superior, inferior and neutral with respect to the Mach cones and is readjusted to the given downstream source intensity. In this general form, the velocities derived from any single source do not seem alarming. If, however, the flux intensities are studied, the fact that all sources have changed to a negative flux density inside the Mach cone, directed toward the source points, and have a not integrable total flux of increasing negative value when approaching the cone walls, will be of a very alarming nature.

The only reassuring feature can be found in the fact, that an array of equal sources spread out along the z axis from infinity to infinity has to simulate the two dimensional source in the x and y plane. Such a supersonic source has already a very peculiar behavior: it sprays half its intensity along the upper side of the Mach wedge and the other half along the lower side, leaving the inside of the downstream Mach wedge completely undisturbed. Anybody who tries to achieve this result with an array of shower nozzles with an axially symmetric spray will be happy to learn the rules of this game by imitating the von Karman source in equation (7). Firstly, the single source touches the wedge only along one generatrix, where no other source can reach. The total intensity at this generatrix must be equal to half the source strength, to fulfill the local requirement now or never. Having an axially symmetric character, the positive source strength along the total outer Mach cone is infinitely too high. The compensation can be sucked back only in a special manner of negative flux inside the Mach cone exactly as the finite intensities indicate in equation (7). Thus the creditability of the equation (7) is easily established as soon as the strange behavior of the two dimensional source is recognized. When superpositioning is equivalent with averaging, a highly extreme, but positive only, distribution can only be the average of an even more extreme distribution of positive and negative contributions.

When the spotlight is put on the strange behavior of three dimensional sources: to spray too much along the Mach cone and to suck it all back except for the little positive flux indicated by the source strength, the historical fact, that the sonic boom had much more intensity than anybody had expected, will appear more justified. Of course, the two dimensional flight would give no relief with distance in the linearized approach; but the attenuation with the negative one half power of the distance in three dimensions means less than the two dimensional flux along a cone surface. Such attenuation results from the fact, that a source intensity spread over a finite length in flight direction will quickly alleviate for the near field. In the far field the relative size of the source spread compared to the distance reduces with distance and this causes the attenuation with less than the minus first power of the distance.

V. The Relation Between the Body Shape and the Far Field Pressure. The actual far field pressure and the body shape can be related by the equation (7) when the source distribution results from the body cross section $A(\xi)$ along the axis of the body. This relation, though the inverse of a given source distribution, is also affected by the odd behavior of the three dimensional sources, to overspray their intensity and to suck the surplus back. For very thin bodies the task to fill the inside with a simulated parallel flow of $\rho_\infty UA(\xi)$ becomes so predominant that the necessary source strength $s(\xi)$ along the axis is sufficiently represented by

$$s(\xi) = \rho_\infty U \frac{dA}{d\xi} = \rho_\infty U A'(\xi) \qquad (8)$$

If this rather simple relation is used in equation (7) for axially symmetric bodies as a starting point, the potential reduces to a single integral along the axis as follows:

$$\varphi(x,y,z) = -\int_{-\infty}^{x} \frac{U A'(\xi)\, d\xi}{2\pi\sqrt{(x-\xi)^2-(M^2-1)(y^2+z^2)}} \tag{9}$$

Such expression is much too discontinuous along the Mach cone as to allow the necessary partial differentiations with respect to x, y, and z for the velocity components. To achieve reasonable pressures it is essential to impress on the designing engineer the idea to make his task, the choice of $A(\xi)$, smoother and smoother. He should watch the existence and limitations of higher derivatives if so desired. Such demand is expressed by the process of integrating the kernel function by parts with respect to ξ and to differentiate correspondingly the area distribution. Two such steps may show the idea:

$$\varphi(x,y,z) = -\int_{-\infty}^{x} \frac{U A''(\xi)}{2\pi} \cosh^{-1} \frac{x-\xi}{\sqrt{(M^2-1)(y^2+z^2)}}\, d\xi \tag{9a}$$

and

$$\varphi(x,y,z) = -\int_{-\infty}^{x} \frac{U A'''(\xi)}{2\pi} \left\{ (x-\xi)\cosh^{-1}\frac{x-\xi}{\sqrt{(M^2-1)(y^2+z^2)}} - \sqrt{(x-\xi)^2-(M^2-1)(y^2+z^2)} \right\} d\xi \tag{9b}$$

The pressure disturbance is in first approximation depending on any velocity component parallel to the main flow U. Such pressure disturbance may be compared to the static pressure p_∞ of the undisturbed flow which is for perfect gases also depending upon the density ρ_∞, the velocity of sound a_∞, and the ratio of the specific heats c_p and c_v with $\gamma = c_p/c_v$:

$$\Delta p = -\varrho_\infty U \frac{\partial \phi}{\partial x} \tag{10}$$

$$p_\infty = \frac{1}{\gamma} \varrho_\infty a_\infty^2 \tag{11}$$

The relative pressure disturbance is now given by (9b) differentiated with respect to x (or by (9a) differentiated one more time on A, which is von Karman's trick to get there fast):

$$\frac{\Delta p}{p_\infty} = \frac{\gamma M^2}{2\pi} \int_{-\infty}^{x} A'''(\xi) \cosh^{-1} \frac{x-\xi}{\sqrt{(M^2-1)(y^2+z^2)}}\, d\xi \tag{12}$$

For finding the pressure of the sonic boom, the inverse cosh function is only used for very small distances ξ in the direction x and very large distances h in y or z; but the distance x, to hear the pressure has to be large enough to be inside the Mach cone which starts at the tip of the aircraft. The proper scale of the noise locations would be better served when the distance X beyond the Mach cone from the tip on the ground is used with $X = x - \sqrt{M^2 - 1}\, h$. The inverse sinh is for small values equal to the argument, while the relation holds: $\sinh^2 = \cosh^2 - 1$. The $\cosh^{-1}$ can thus be converted for small deviations of its argument from 1 into an algebraic equivalent:

$$\frac{\Delta p}{p_\infty} \approx \frac{\gamma M^2}{2\pi} \int_{-\infty}^{x} A'''(\xi) \sqrt{\frac{(x-\xi)^2}{(M^2-1)h^2} - 1}\, d\xi \approx \frac{\gamma M^2 \sqrt{2}}{2\pi \sqrt[4]{(M^2-1)} \sqrt{h}} \int_{0}^{X} A'''(\xi) \sqrt{X-\xi}\, d\xi \tag{13}$$

Equation 13 shows explicitly the attenuation of the far field pressure with the inverse half power of the distance h, while the other expressions indicate an otherwise frozen pressure footprint as soon as the cross section distribution $A(\xi)$ is assumed.

VI. The Influence of a Stratified Atmosphere. The linear differential equation (3) expects a homogeneous atmosphere and results as such in equation (13) for a lateral observer at a distance h. Actually the flight takes place in vertical direction and the stratified atmosphere is a major factor when assessing the intensity of the created boom. The adaptation of the former result to the density changes at almost constant temperature is not difficult to indicate. The acoustical response or the impedance of the stratified atmosphere can be investigated by assuming finite changes in successive layers. Their discontinuities are able to reflect a part of the incoming signal and to let another part of the signal go on. A softer medium reflects partially with reversed pressures, a harder medium with pressures of the same sign. The resulting change of the continuing wave inside the new layer is the sum of the incoming and the reflected pressures. The adaptation of the surface velocities can be achieved by the fact, that the reflected wave, going in the opposite direction, has the reverse relations between the pressures and the velocities compared to both the incoming and the continuing waves. The extreme values of infinitely hard or soft reveal the arithmetic progress of double or nothing in either pressure or velocity, but the small local changes result in a geometric progress relation, half the impedance change charged to the pressure increase and half to the velocity decrease in a logarithmic scale. All continuous changes can be handled in this manner, discontinuous changes give finite intensities of the reflections and would be more desirable to attenuate the continuing wave. The only other question is about any rereflections which will also be created and may follow a short signal after a while or may modify a longer signal under its own rereflections. The exponentially stratified atmosphere with a large scale height can be released from this rereflection trouble since they are either too small or too late to modify the original signal. According to the remaining adaptation of the original signal according to the square root of the pressure changes in the atmosphere, equation (13) can be corrected for both the gradual change of the static pressure in the atmosphere with p_∞ at the altitude h and P at the surface of the earth with

$$\frac{p_\infty}{P} = \exp\left(-\frac{h}{H}\right) \tag{14}$$

Taking also into account the doubling of the wave pressure on the solid surface of the earth with $\Delta P = 2\,\Delta p$, the effective influence of the two effects is a factor 2 and a remaining relief of the square root of p_∞/P since the original relief of moving a body in lower density is half lost on the way back to earth by adapting the disturbance gradually to denser air:

$$\frac{\Delta P}{P} = \frac{\gamma M^2 \sqrt{2}}{\pi \sqrt[4]{M^2-1}\,\sqrt{h}} \exp\left(-\frac{h}{2H}\right) \int_0^X A'''(\xi)\sqrt{X-\xi}\, d\xi \tag{15}$$

Both functions of h are indicating relief when an aircraft body is moved at higher altitude h, though even their combined effect is much less than would be desirable.

VII. Relations Between Lift and Pressure. Lift on wings is generally explained by bound vortices inside the wings and by trailing vortices in the flow direction. It seems, therefore, that the whole development of singularities and their fields has to be repeated for vortices instead of sources and sinks. Fortunately this is not necessary since from two-dimensional incompressible flow the identity of a pair of vortices with a pair of sources and sinks normal to their direction is well known. The trailing vortices following the flow direction create their field normal to the x axis and, therefore are unaffected in this identity by compressibility. Since the lift-contribution of the bound vortex is the dipole moment of its trailing vortex pair multiplied with $\rho_\infty U$, the potential equation for distributed sources (7) can be changed to a supersonic normalized potential for any lift distribution $\ell(\xi,\eta,\zeta)$. It is only necessary to integrate the potential of the source from any given position to infinity in x direction as an array of sources and then to differentiate the result with respect to ζ to create source and sink dipoles normal to the trailing vortices. The former source distribution $s(\xi,\eta,\zeta)$ must be replaced in the new potential by $\ell(\xi,\eta,\zeta)/U$. Integrations with respect to ξ are already used in equations (9a) and (9b), the only new operation is the differentiation with respect to ζ and gives:

$$\varphi(x,y,z) = -\iiint \frac{\ell(\xi,\eta,\zeta)\,(x-\xi)(z-\zeta)\;d\xi\,d\eta\,d\zeta}{2\pi\rho_\infty U\sqrt{(x-\xi)^2-(M^2-1)[(y-\eta)^2+(z-\zeta)^2]}\,\left[(y-\eta)^2+(z-\zeta)^2\right]} \qquad (16)$$

For the far field the distribution of the lift in y direction over the wing span is immaterial and the distribution in z direction with dihedral or biplanes may be studied separately; thus the simple case of a wing in the x,y plane has a certain lift contribution $\ell(\xi)$ counted along the x axis, which can be integrated up to the total lift $L(\xi)$ with $\ell(\xi) = L'(\xi)$ and the simplified lift potential reads:

$$\varphi(x,y,z) = -\int_{-\infty}^{x} \frac{L'(\xi)\,(x-\xi)\,z\;d\xi}{2\pi\rho_\infty U\sqrt{(x-\xi)^2-(M^2-1)(y^2+z^2)}\,\left[y^2+z^2\right]} \qquad (17)$$

There are similar integrations by parts available for the lift distribution as for the area distribution of the body to allow the differentiation with respect to x, y, or z without degenerations when any component of the disturbance velocities u, v, and w are needed.

$$\varphi(x,y,z) = -\int_{-\infty}^{x} \frac{\sqrt{M^2-1}\;z\,L''(\xi)}{2\pi\rho_\infty U\sqrt{y^2+z^2}}\sqrt{\frac{(x-\xi)^2}{(M^2-1)(y^2+z^2)}-1}\;d\xi \qquad (17a)$$

$$\varphi(x,y,z) = -\int_{-\infty}^{x} \frac{\sqrt{M^2-1}\; z\; L'''(\xi)}{4\pi \rho_\infty U \sqrt{y^2+z^2}} \left\{ (x-\xi)\sqrt{\frac{(x-\xi)^2}{(M^2-1)(y^2+z^2)} - 1} - \sqrt{(M^2-1)(y^2+z^2)}\, \cosh^{-1} \frac{(x-\xi)}{\sqrt{(M^2-1)(y^2+z^2)}} \right\} d\xi \quad (17)$$

Since for the pressure distribution only the component u is of interest and can be derived from the potential by one more derivation of L, the equation (17a) without difficult derivations leads to the disturbance pressure $\Delta p = -\rho_\infty Uu$:

$$\frac{\Delta p}{p_\infty} = \frac{\sqrt{M^2-1}}{2\pi\, p_\infty} \int_{-\infty}^{x} L'''(\xi) \sqrt{\frac{(x-\xi)^2}{(M^2-1)h^2} - 1}\; d\xi \quad (18)$$

In (18) both sides are divided by p_∞ to demonstrate the close relations between the pressure due to lift and the pressure due to body shape in equation (12). As the equation (13) shows in its first part, the far field approximations for both body and lift are only different in the constants outside the integral and permit the second simplification of (13) for the lift:

$$\frac{\Delta p}{p_\infty} = \frac{\sqrt[4]{M^2-1}\,\sqrt{2}}{2\pi\, p_\infty \sqrt{h}} \int_{0}^{X} L'''(\xi) \sqrt{X-\xi}\; d\xi \quad (19)$$

With such close resemblance the adaptation to the stratified atmosphere is easy to perform. The doubling by reflection on the ground remains the same. Only the fact that the lift was given by weight regardless of the pressure p_∞ of the altitude, adds the ratio p_∞/P in the denominator and reverses, therefore, the sign of the half power exp(-h/2H) to exp(h/2H):

$$\frac{\Delta P}{P} = \frac{\sqrt[4]{M^2-1}\,\sqrt{2}}{\pi P \sqrt{h}} \exp\left(\frac{h}{2H}\right) \int_{0}^{X} L'''(\xi) \sqrt{X-\xi}\; d\xi \quad (20)$$

The final equation for the lift footprint by the linearized theory is very similar to the equation (15) with respect to the integral over the design parameters A respectively L. The difference is, however, that A must disappear when the total aircraft length is used, while the lift remains finite and has to equal the weight. An analogy is more directly between the body nose and the total lift. Body nose and lift are thus rivals with respect to the permissible pressure footprint, while body tail and lift reduce their pressures when superimposed. The altitude dependency of the footprint pressure is for lift a limited attenuation reached at $h = H$ or about 28,000 feet. For the body part of the pressure footprint any higher altitude is still beneficial for the pressure directly under the flight path, though, of course, spreading laterally the permissible pressure proportional to the altitude.

VIII. Formation of Shock Waves. The footprint of the linearized theory both for body shape and for lift is integrated toward the far field beyond its proper validity. A reduced pressure disturbance would be necessary to avoid such error. We may, however, investigate the combination of a first small pressure plateau wide enough to let a second pressure wave ride on it piggyback. There are two reasons to make the second wave ride faster on this plateau than on the undisturbed air. The pressure step Δp creating the first plateau has provided along the plateau a forward velocity u which adds to the second wave speed. There is also a slight temperature increase connected with the formation of a pressure plateau Δp which for all perfect gases is indicative of a higher speed of sound. The wave speed increment ΔW is, therefore, composed of two contributions u and Δa both of them simply related to the pressure step Δp compared with the static pressure $p_\infty = \frac{1}{\gamma} \rho_\infty a_\infty^2$ and with the original wave speed a_∞. The disturbance velocity u is simply $\frac{\Delta p}{\rho_\infty a_\infty}$ while the isentropical temperature change is the $\frac{\gamma-1}{\gamma}$ power of the pressure change and the velocity of sound is the square root of the temperature change. The relative increase of the wave speed is, therefore:

$$\frac{\Delta W}{a_\infty} = \frac{u}{a_\infty} + \frac{\Delta a}{a_\infty} = \left(\frac{1}{\gamma} + \frac{\gamma-1}{2\gamma}\right)\frac{\Delta p}{p_\infty} = \frac{\gamma+1}{2\gamma}\frac{\Delta p}{p_\infty} \tag{21}$$

The result from this investigation is that the different pressure levels have an excess wave speed proportional to their own pressure level. The different phases of the footprint except for the near field have a similar history on the way down, the total progress of any elevated pressure on the ground can be integrated and is in the integral still proportional to the pressure on the ground. A common ratio $\Delta X/\Delta P$ exits therefore and can easily be applied to rectify the positions of the linearized footprint pressures. Since the sound waves propagate perpendicular to the Mach wave, the descent path of the wave is larger than the altitude h by $1/\cos\alpha$ with α the Mach angle difined by $\sin\alpha = 1/M$. The visible advance toward the front $-\Delta X$ on the ground is also larger than the actual progress achieved by excess wave speed, since the advanced Mach wave front is inclined and needs only $-\Delta X \sin\alpha$ progress to appear as $-\Delta X$ measured along the ground. With these geometrical factors, the length $-\Delta X$ of the advancing waves must be integrated according to the wave strength divided by the static pressure at every distance z over the whole altitude range h on the descent dz of the wave:

$$-\frac{\Delta X}{\Delta P} = \frac{\gamma+1}{2\gamma}\frac{M^2}{\sqrt{M^2-1}}\frac{h}{2P}\int_0^h \exp\left(\frac{h-z}{2H}\right)\frac{dz}{\sqrt{zh}} = \frac{\gamma+1}{2\gamma}\frac{M^2}{\sqrt{M^2-1}}\frac{h}{P}\frac{\sqrt{\pi}}{2}\exp\frac{h}{2H}\frac{\operatorname{erf}\left(\sqrt{h/2H}\right)}{\sqrt{h/2H}} \tag{22}$$

The error function starts linear but never exceeds the value unity. For moderate altitudes h of one to three scale heights H the error function over its argument and the factor $\sqrt{\pi}/2$ may be approximated as another exponential function diminishing with the argu ment - h/8H making the whole bracket an exponential with 3h/8H just for the sake of simplicity. This approximation reads:

$$-\frac{\Delta X}{\Delta P} \approx \frac{h}{P}\frac{\gamma+1}{2\gamma}\frac{M^2}{\sqrt{M^2-1}}\exp\frac{3h}{8H} \tag{22a}$$

To compensate for the progress of positive pressure disturbances toward the airplane tip or beyond, which means a negative shift in X as the equations 22 and 22a indicate, the ordinate axis can be slanted in the positive direction to interprete both the footprints for body shape (15) and lift (20) correctly. A complete aircraft is, of course, a superposition of body and wing and is the sum of (15) and (20) before the proper shock waves can be found under the positive slope of equation (22a)(with reversed sign) and by the equal area exchange left and right from every shock wave. If the permissible disturbance pressure is one thousandth of the atmospheric pressure and the altitude is 8H/3 = 70,000 feet, the progress $-\Delta X$ for a Mach number of 2.5 and $\gamma = 1.4$ amounts to 450 feet or too much more than the aircraft length. Coming down to 35,000 feet altitude reduces the length $-\Delta X$ to 135 feet which is still too large a portion of aircraft length for the nose. It is obvious that the formation of shocks can easily lengthen for flights at high altitudes the distance between the two booms to multiples of the aircraft length. The opposite attempt, to avoid shocks, spends large portions of the aircraft length just for softening the footprint. All under the assumption that a disturbance pressure of one thousandth of the atmosphere is feasible and permissible.

IX. Applications. The wave patterns created by aircraft bodies and wings are given in integral form in equations (15) and (20). The physical input is such a well behaved parabola $\sqrt{X-\xi}$ that the design input A''' or L''' does not have to be smooth, but can be concentrated on very few centers where their integrals correspond to discontinuities in the next higher function $\Delta A''$ or $\Delta L''$ respectively. The only visible mark in the pressure curve on such places is the infinite slope of any new parabola on its apex which can be neglected when the summation is used as a fast substitute for exact integration; but if it is real, it does indicate a shock on positive rises wherever the design is actually made with sudden changes in the curvature of A (ξ) or L (ξ). To see whether such changes in curvature are permissible and in which order of magnitude they may cause shocks, a conical tip of the body and the

~~and th~~e tip of a delta wing with constant load may be investigated under the condition that the parabolic pressure of the linearized theory is corrected to a shock wave with a parabolic continuation according to equation (22a) and that the shock wave intensity is just one thousandth of the atmosphere on the ground. In the equations (15) and (20) the quantities in question on the tip $\xi = 0$ are $\Delta A''(0)$ and $\Delta L''(0)/P$; both are in this form dimensionless, while the factor $\sqrt{X-0}$ finds its partner in $\sqrt{h}$ to nondimensionalize the whole equations. It is possible to find the proper order of magnitude by identifying $-\Delta X$ from equation (22a) with $X - 0 = X$ in the other equations, which indicates the point on the parabola that moves to the tip Mach cone; but the exact shock wave moves somewhat ahead of the tip Mach cone and requires $-\Delta X = 4/3\ X$, when X identifies exactly the location of the pressure at the top of the shock wave along the linearized parabola. By matching $\sqrt{-3/4\,\Delta X/h}$ from equation (22a) with $\sqrt{X/h}$ either from (15) or (20) and by inserting the relative disturbance pressures $\Delta P/P$ the permissible value, the unknown quantities $\Delta A''(0)$ respectively $\Delta L''(0)$ will be fixed:

$$\sqrt{-\frac{3}{4}\frac{\Delta X}{h}} = \sqrt{\frac{\Delta P}{P}\,\frac{3(\gamma+1)}{8\gamma}\,\frac{M^2}{\sqrt{M^2-1}}\exp\frac{3h}{8H}} = \frac{\Delta P}{P}\,\frac{\pi\sqrt[4]{M^2-1}}{\gamma M^2\sqrt{2}\,\Delta A''}\exp\frac{h}{2H} = \sqrt{\frac{X}{h}} \qquad (23)$$

and

$$\sqrt{\frac{\Delta P}{P}\,\frac{3(\gamma+1)}{8\gamma}\,\frac{M^2}{\sqrt{M^2-1}}\exp\frac{3h}{8H}} = \frac{\Delta P}{P}\,\frac{\pi}{\sqrt[4]{M^2-1}\sqrt{2}\,(\Delta L''/P)}\exp\left(-\frac{h}{2H}\right) \qquad (24)$$

These relations furnish:

$$\Delta A''(0) = \sqrt{\frac{\Delta P}{P}}\,\frac{2\pi\sqrt{M^2-1}}{M^3\sqrt{3\gamma(\gamma+1)}}\exp\frac{5h}{16H} \qquad (23a)$$

and

$$\frac{\Delta L''(0)}{P} = \sqrt{\frac{\Delta P}{P}}\,\frac{2\pi}{M\sqrt{3(\gamma+1)/\gamma}}\exp\left(-\frac{11}{16}\frac{h}{H}\right) \qquad (24a)$$

The actual area distribution or lift distribution permissible with respect to the assumed shock strength of 10^{-3} atm with P = 1 atm = 2116 lb/ft^2 for M = 2.5

$$A(\xi) = \frac{\Delta A''(0)}{2}\,\xi^{-2} = 0.0046\exp\left(\frac{5h}{16H}\right)\xi^{2} \qquad (25)$$

and

$$L(\xi) = \frac{\Delta L''(0)}{2}\,\xi^2 = 37\,\frac{\text{lb}}{\text{ft}^2}\,\exp\left(-\frac{11}{16}\frac{h}{H}\right)\xi^2 \tag{26}$$

Both these equations demonstrate, that the modern supersonic transport projects are in the proper order of magnitude; but they also indicate how difficult the design would be, if the shock strength has to be reduced to 10^{-4} atmospheres. A cross section of 200 ft.^{2} is hardly reached in a conical tip of ξ=200 ft. length according to (25), and 500,000 lb. weight require the length of more than 120 ft. in a delta wing in (26), unless the tail section of the fuselage can compensate as a sink the source character of the lift toward the ground.

Fig 1. Slanted Ordinate Axis

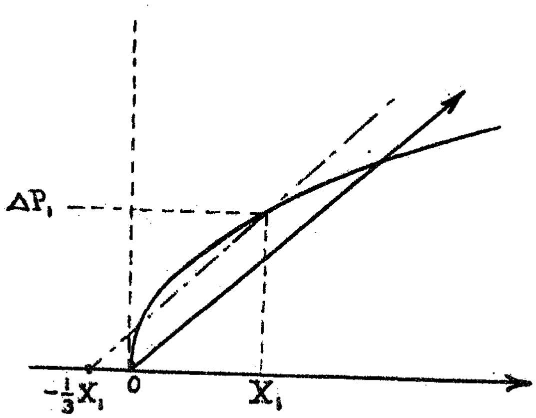
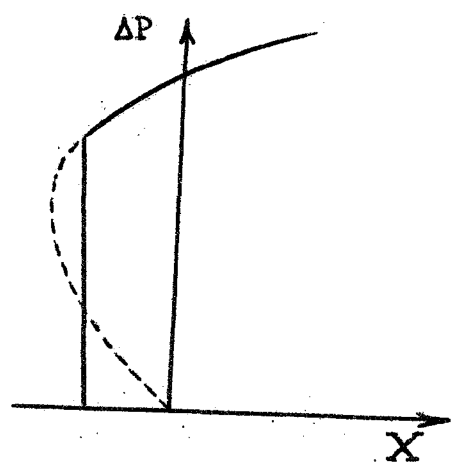


Fig 2. Corrected Pressure Diagram



Suggested Further Reading

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